

**Phys 410**  
**Fall 2013**  
**Lecture #13 Summary**  
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Constrained systems are common in physics, and their dynamics can be advantageously solved by the Lagrangian method. Examples include the pendulum, the Atwood machine, a rigid body, gas particles in a box, a rolling object, and a bead on a wire. We considered the pendulum problem in detail. The constraint is that the length of the pendulum  $\ell$  is fixed, so that the x- and y-coordinates of the bob are not independent, but constrained so that  $\ell = \sqrt{x^2 + y^2}$ . We can incorporate this constraint by adopting a new independent variable to describe the position of the bob, namely the angle that the pendulum makes with the vertical,  $\phi$ . In terms of this generalized coordinate, the Lagrangian becomes  $\mathcal{L}(\phi, \dot{\phi}) = \frac{m}{2} \ell^2 \dot{\phi}^2 - mg\ell(1 - \cos \phi)$ . Lagrange's equation gives  $-mg\ell \sin \phi = m\ell^2 \ddot{\phi}$ , which relates the torque due to gravity on the bob to the time rate of change of the angular momentum of the bob, or the moment of inertia ( $m\ell^2$ ) times the angular acceleration ( $\ddot{\phi}$ ). Note that the force of constraint (namely the tension in the rod supporting the bob) never played a role in the analysis (whereas it plays an important role in the traditional Newtonian approach). Once the appropriate generalized coordinate is identified, the associated constraining force disappears from the discussion!

Generalized coordinates and constrained systems are important for Lagrangian dynamics. Consider a system consisting of  $N$  particles, with positions  $\vec{r}_\alpha$ , with  $\alpha = 1, \dots, N$ . The parameters  $q_1, q_2, \dots, q_n$  are a set of generalized coordinates if each position  $\vec{r}_\alpha$  can be expressed as a function of  $q_1, q_2, \dots, q_n$ , and possibly time  $t$  as,  $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \dots, q_n, t)$  for  $\alpha = 1, \dots, N$ , and the inverse  $q_i = q_i(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t)$  for  $i = 1, 2, \dots, n$  can also be written. For particles in three dimensions,  $n \leq 3N$ . If  $n < 3N$ , then the system is said to be constrained. The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. The simple pendulum is constrained and has one degree of freedom. The double pendulum is constrained and has two degrees of freedom. One can show (the proof is in Taylor) that constrained systems obey the Lagrange equations when their Lagrangian is written in terms of the generalized coordinates of the system.

We then did the example of the Atwood machine for a frictionless and inertia-less pulley supporting two different masses. The masses can each move in one dimension (which we called  $x$  and  $y$ ), and their motion is constrained because they are on either end of a string of fixed length. The constraint is that the string length is  $\ell = x + y + \pi R$ , where  $R$  is the radius of the pulley. With this constraint incorporated, the Lagrangian can be written as  $\mathcal{L}(x, \dot{x}) =$

$\frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + \text{const.}$  Note that the constant plays no role in the dynamics since it disappears when both of the derivatives  $(\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial \dot{x}})$  are taken. The resulting equation of motion is  $\ddot{x} = g \frac{m_1 - m_2}{m_1 + m_2}$ . Again note that the constraining force (the tension in the string) was never mentioned or considered in the process. The tension is essential to the traditional Newton's second law approach to solving this problem.

We concluded by looking at the pendulum whose point of suspension is forced to rotate on a circle of radius  $R$  at a fixed angular velocity  $\omega$ . The key step is to write down the  $(x, y)$  coordinates of the bob in terms of a minimum number of parameters and generalized coordinates. We did this by describing the location of the bob starting from the center axis of the circle (chosen to be the origin) and describing the location of the point of suspension, and then adding the vector position of the bob relative to the point of suspension. The location of the particle is specified by a single variable,  $\varphi$ , which describes the deviation of the bob from the vertical. This location  $(x, y)$  was then differentiated with respect to time to get the vector velocity (in terms of  $\varphi$ ,  $\dot{\varphi}$ , and time  $t$ ), and the kinetic energy was constructed from that. The potential energy is entirely due to gravity, so the Lagrangian can be constructed. The Euler-Lagrange equation yields the equation of motion for the single generalized coordinate  $\varphi$ . The resulting motion can be quite complicated.